tion rates changing with time of membrane exposure to hydrogen was due to a surface effect. In fact the increase of permeation rates observed under identical conditions of pressure and temperature can be considered to be the result of an increase in surface activity.

These results, inadvertently obtained here, showing that the permeation rates changed with time, support the evidence that surface reactions exert a great influence on permeation rates in the hydrogen type 321 stainless steel system. However they tended to hinder quantitative treatment of the data. For example the proposed permeability expression should be considered valid only after any initial activation of the membrane surface has taken place.

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NOTATION

A = area, permeation area, area perpendicular to the direction of gas flow, sq. cm.

A_a = arithmetic average permeation area for a hollow cylinder, sq. cm.

c = concentration (atomic) of gas within the metal lattice, cc. (NTP)/cc.*

D = diffusivity or diffusion coefficient, sq. cm./sec., or sq. cm./min.

D_o = value of D at infinite temperature, frequency factor, sq. cm./min.

E_d = activation energy for diffusion, cal./g. mole

E_p = activation energy for the permeation process, cal./g. mole

E_s = activation energy for solubility or heat of solution, cal./g. mole

h = mass transfer coefficient for interfacial-resistance model, cc.(NTP)/sq. cm. min.(atm.)$^{1/2}$*

K = solubility constant, cc.(NTP)/cc. (atm.)$^{1/2}$

K_o = solubility constant at infinite temperature, cc. (NTP)/cc. (atm.)$^{1/2}$

L = length of permeation membrane, cm.

n = pressure exponent, $1/2 < n < 1$

P = permeability, $\left[\frac{\text{cc. (NTP)}}{\text{sq. cm. min.}}\right]\left(\frac{\text{mm.}}{(\text{atm.})^{1/2}}\right)$

P_o = permeability at infinite temperature, $\left[\frac{\text{cc. (NTP)}}{\text{sq. cm. min.}}\right]\left(\frac{\text{mm.}}{(\text{atm.})^{1/2}}\right)$

p = gas pressure, atm.*

R = rate of permeation, total rate of gas permeation or diffusion through a metal barrier, cc. (NTP)/min.

R = molar gas constant, 1.987 cal./g. mole (°K.)

r = radial distance in a cylinder, cm.*

r_a = arithmetic average radius, equal to r_m for thin-walled cylinders, cm.

r_m = log mean radius, $(r_2 - r_1)/\ln r_2/r_1$, cm.

T = absolute temperature, °K.

* Subscripts 1 and 2 refer to conditions at the entrance and exit surfaces, respectively, for p, c, r, and h.

LITERATURE CITED

1. Armbruster, M. H., *J. Am. Chem. Soc.*, **65**, 1043 (1943).
2. Ash, R., and R. M. Barrer, *Phil. Mag.*, **47**, 1197 (1959).
3. Barrer, R. M., "Diffusion in and through Solids," Cambridge University Press, Cambridge, England (1951).
4. Baukloh, W., and H. Kayser, *Z. Metallk.*, **26**, 156 (1934).
5. Bryan, W. C., D. Eng. dissertation, Yale Univ., New Haven, Connecticut (1959).
6. Chang, P. L., and W. D. G. Bennett, *J. Iron and Steel Inst.*, **170**, 208 (1952).
7. Flint, P. S., Knoll's Atomic Power Laboratory, Schenectady, New York, *KAPL-659* (December 14, 1951).
8. Harden, K. L., M. Eng. thesis, Yale Univ., New Haven, Connecticut (1957).
9. Johnson, F. M. G., and P. Larose, *J. Am. Chem. Soc.*, **46**, 1377 (1924).
10. Lewkonja, G., and W. Baukloh, *Z. Metallk.*, **25**, 399 (1933).
11. Lombard, V. M., *Compt. rend.*, **177**, 116 (1923).
12. "Metals Handbook," Am. Soc. for Metals, Cleveland, Ohio (1948).
13. Phillips, J. R., D. Eng. dissertation, Yale Univ., New Haven, Connecticut (1960).
14. Raczynski, W., *Arch. Hutnictwa*, **3**, 59 (1958). (See *Chem. Abs.*, **52**, 15205.)
15. Richardson, O. W., *Phil. Mag.*, **7**, 266 (1904); **8**, 1 (1904).
16. Schenk, H., and H. Taxhet, *Arch. Eisenhüttenw.*, **30**, 661 (1959).
17. Silberg, P. A., and C. H. Bachman, *J. Chem. Phys.*, **29**, 777 (1958).
18. Sims, C. E., "Gases in Metals," pp. 119-198, Am. Soc. for Metals, Cleveland, Ohio (1953).
19. Tetelman, A. S., Private communication, Yale Univ., New Haven, Connecticut (December 29, 1959).
20. Wahlin, H. B., and V. O. Naumann, *J. Appl. Phys.*, **24**, 42 (1953).

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Behavior of Non-Newtonian Fluids in the Inlet Region of a Channel

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Under the impetus of both academic curiosity and practical necessity considerable effort has been expended in recent years toward finding solutions to the differential equation of motion for non-Newtonian fluids. As one might expect exact solutions are available only for relatively simple geometries and rheological equations of state (the set of equations relating the state of stress of the fluid under consideration to its velocity field). For a discussion dealing with some of the problems inherent in obtaining solutions to the equation of motion the reader is referred to a review by Oldroyd (7).

It has become evident that approximate solutions to the equation of motion can be of value for non-Newtonian

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fluids in the same manner that approximations have been of vital importance in advancement of Newtonian fluid mechanics. Thus the use of boundary-layer theory for non-Newtonian systems has been discussed by Schowalter (*9*) and specific applications have been reported by Acrivos and coworkers (*1*, *2*). In the present paper principles of boundary-layer theory have been applied to determine the behavior of a non-Newtonian fluid in the entry region of a channel. The results apply to laminar flow of pseudoplastic fluids whose rheological behavior can be approximated by a power-law relation between shear stress and velocity gradient.

Earlier workers (*4*, *11*) have used approximate forms of boundary-layer theory to estimate the entry length for axisymmetric pipe flow. The treatment described below differs from these in two respects. A different geometry is considered, and a different treatment of the boundary layer is utilized. The results provide estimates of the pressure drop which occurs in the entry region, the length of channel required to establish fully developed laminar flow, and the shape of velocity profiles in the entry region. The technique used is an extension of the approach employed by Schlichting (*10*) for Newtonian fluids. It is believed that the results provide a better description of velocity profiles in the entry region than would have been possible through application of methods previously used for axisymmetric pipe flow. The results should be helpful to persons interpreting pressure loss and heat transfer data taken under such conditions that the shape of velocity profile is affected by proximity of the channel entrance.

THEORY

Consider a channel formed by two horizontal semi-infinite plates spaced a distance $2a$ apart. Distances are measured with respect to a Cartesian coordinate system with origin at the end of the lower plate. The x and y axes are horizontal and vertical, respectively, and the z axis is parallel to the edges of the plates. The flow is assumed to be two dimensional ($w = 0$) and to enter the channel at $x = 0$ with a flat velocity profile ($u = U_o$, $v = 0$). Near the channel entry the effect of viscosity of the fluid is important only in the region near the walls, and the velocity profile in the center of the channel remains quite flat, though it has a value greater than U_o because of the deceleration of fluid near the walls. Thus in the region near the entry boundary-layer techniques are applicable. On the other hand at values of x approaching the fully developed region the velocity profile may be described in terms of perturbations on the fully developed flow. The two methods of solution are joined at an appropriate value of x.

The analysis applies to fluids which can be characterized by Bird's generalization of the power law for two-dimensional flow of incompressible fluids (*3*):

$$\mu_{eff} = K\left(\frac{1}{2}I_2\right)^{\frac{n-1}{2}} \qquad (1)$$

where

$$I_2 = \sum_{\beta}\sum_{\alpha}\left[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right]\left[\frac{\partial u_\beta}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_\beta}\right]$$

K and n are empirical constants and the quantity μ_{eff} is defined by a rheological equation of state

$$\tau_{ij} = -\mu_{eff}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (2)$$

Solution Near the Entry (Downstream Solution)

For two-dimensional flow of an incompressible fluid described by Equations (1) and (2) it has been shown that the boundary-layer equation for the x component of velocity may be written (*9*)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \frac{K}{\rho}\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial y}\right\} \qquad (3)$$

The validity of Equation (3) is based on the assumption that the velocity u is a function of x alone over all but a portion δ of the channel near the two walls, and that $\frac{\delta}{a} << 1$. In that portion of the channel where $\frac{\delta}{a} << 1$ it is desired to solve Equation (3) for $u(x, y)$. This may be done through conversion of Equation (3) into an ordinary differential equation.

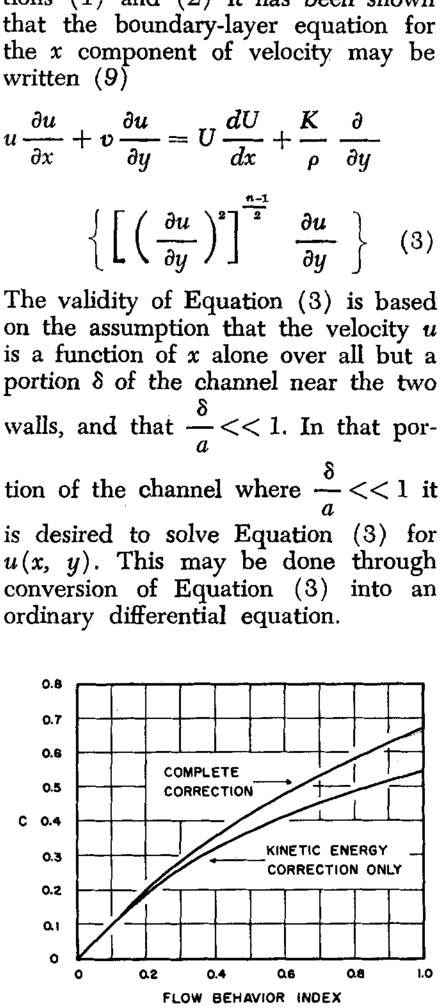

Fig. 1. Entry pressure drop correction as a function of flow behavior index.

Define a stream function such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad (4)$$

and

$$\psi = U_o a\left[\epsilon f_0(\eta) + \epsilon^2 f_1(\eta) + \epsilon^3 f_2(\eta) + \ldots\right] \qquad (5)$$

where $\eta = y\left[\frac{\rho U_o^{2-n}}{Kx}\right]^{\frac{1}{n+1}}$

$$\epsilon = \left[\frac{Kx}{a^{n+1}U_o^{2-n}\rho}\right]^{\frac{1}{n+1}}$$

Then from Equations (4) and (5)

$$u = U_o\left[f_0'(\eta) + \epsilon f_1'(\eta) + \epsilon^2 f_2'(\eta) + \ldots\right] \qquad (6)$$

$$v = \frac{U_o a}{(n+1)x}\left[\eta(\epsilon f_0' + \epsilon^2 f_1' + \epsilon^3 f_2' + \ldots) - \epsilon(f_0 + 2\epsilon f_1 + 3\epsilon^2 f_2 + \ldots)\right] \qquad (7)$$

The velocity $U(x)$ outside of the boundary layer may be expressed through the displacement boundary-layer thickness, where

$$\delta^* = \int_0^a \left(1 - \frac{u}{U}\right)dy \qquad (8)$$

From the continuity equation

$$\int_0^a u\,dy = U_o a \qquad (9)$$

Equations (8) and (9) may be combined to give

$$U(x) = U_o\left[1 + \frac{\delta^*}{a} + \left(\frac{\delta^*}{a}\right)^2 + \ldots\right] \qquad (10)$$

Since δ^* is a function of x alone, one can write

$$U(x) = U_o[1 + K_1\epsilon + K_2\epsilon^2 + \ldots] \qquad (11)$$

By combining Equations (3), (6), (7), and (11) one arrives at a differential equation containing terms made up of coefficients which are functions of η alone multiplied by ϵ raised to different powers (0, 1, 2, . .). When coefficients of like powers of ϵ are equated, an infinite set of ordinary differential equations is obtained, the first one being

$$n(n+1)f_0''' + f_0(f_0'')^{2-n} = 0 \qquad (12)$$

with boundary conditions formed from the requirements of no slip at the wall and $u = U$ at $\eta = \infty$ (considering only one-half of the channel):

$$f_0(0) = f_0'(0) = 0$$

$$f_0'(\infty) = 1$$

Strictly speaking the boundary condition at infinity is inconsistent with the assumption of a uniform velocity near

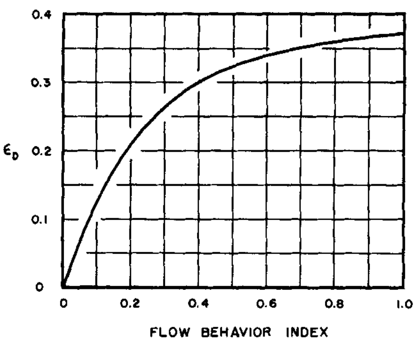

Fig. 2. Entry length as a function of flow behavior index.

the center of a channel of finite width. However in the region where boundary-layer equations are applied the effective boundary layer is sufficiently thin so that difficulties are avoided. Coefficients of higher powers of ϵ yield equations for f_1 through f_5. These equations are given in the Appendix.*

In order to solve Equations (12) and (A1) through (A5) one must have numerical values for the constants $K_1, K_2, \ldots$. Very near the entry the boundary-layer thickness is extremely small. Consequently U is only slightly greater than U_0, and to a good approximation one may terminate the right-hand side of Equation (10) after the second term:

$$U(x) \cong U_0\left[1+\frac{\delta^*}{a}\right]$$

Likewise from Equation (11)

$$U(x) = U_0[1+K_1\epsilon]$$

or

$$\delta^* = aK_1\epsilon$$

But since the entry is near, one has from Equations (6) and (8)

$$\delta^* \cong \int_0^a [1-f_0']\,dy = a\epsilon\,[\eta_1 - f_0(\eta_1)]$$

where η_1 is some value of η outside of the boundary layer. Then

$$K_1 = \eta_1 - f_0(\eta_1) \qquad (13)$$

and f_0 may be found from a solution of Equation (12), which is just the solution for flow past a flat plate (*1*).

Slightly farther downstream one must retain three terms in the right-hand side of Equation (10). A procedure similar to that outlined above leads to

$$\delta^* \cong \int_0^a \left[1-\left(\frac{f_0'+\epsilon f_1'}{1+K_1\epsilon}\right)\right] dy$$

Upon evaluation of the integral one finds

$$K_2 = K_1\eta_1 - f_1(\eta_1) \qquad (14)$$

and f_1 may be found from solution of Equation (A1). Similarly

$$K_i = K_{i-1}\,\eta_1 - f_{i-1}(\eta_1) \qquad (15)$$

In principle as many ordinary differential equations of the type Equations (12) and (A1) through (A5) may be formed as is desired. The present work employs six equations for f_0 through f_5. Knowing f_0 through f_5 one can compute the velocity in the boundary layer by writing Equation (6) in the form

$$u \cong U_0\,[f_0' + \epsilon f_1' + \epsilon^2 f_2' + \epsilon^3 f_3' + \epsilon^4 f_4' + \epsilon^5 f_5'] \qquad (16)$$

Solution Approaching Fully Developed Flow (Upstream Solution)

Now consider the equation of motion in the region of the channel which approaches fully developed flow. Since $\frac{\partial^2 u}{\partial y^2} >> \frac{\partial^2 u}{\partial x^2}$, the equation of motion may be written

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial Y}\int_Y^1 \left(\frac{\partial u}{\partial x}\right) dY = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{K}{\rho a^{n+1}}\frac{\partial}{\partial Y}\left\{\left[\left(\frac{\partial u}{\partial Y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial Y}\right\} \qquad (17)$$

where $\quad Y = 1 - \frac{y}{a}$

and the continuity equation has been used in the form

$$\frac{\partial u}{\partial x} - \frac{1}{a}\frac{\partial v}{\partial Y} = 0 \qquad (18)$$

Making the reasonable approximation that $p = p(x)$, and differentiating Equation (17) partially with respect to Y, one obtains

$$u\frac{\partial^2 u}{\partial x \partial Y} + \frac{\partial^2 u}{\partial Y^2}\int_Y^1 \frac{\partial u}{\partial x}\,dY = \frac{K}{\rho a^{n+1}}\frac{\partial}{\partial Y}\left\{\frac{\partial}{\partial Y}\left\{\left[\left(\frac{\partial u}{\partial Y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial Y}\right\}\right\} \qquad (19)$$

If one considers the velocity u to be expressed as a perturbation of the fully developed profile, it is found that

$$u = U_0\left(\frac{2n+1}{n+1}\right)\left(1 - Y^{\frac{n+1}{n}}\right) + u^* \qquad (20)$$

Substitution of Equation (20) into (19) results in

$$\left(\frac{2n+1}{n+1}\right)\left[1 - Y^{\frac{n+1}{n}}\right]\frac{\partial^2 u^*}{\partial x \partial Y} - \left(\frac{2n+1}{n^2}\right) Y^{\frac{1-n}{n}}\int_Y^1 \frac{\partial u^*}{\partial x}\,dY = \frac{K(2n+1)^{n-1}}{U_0^{2-n} n^n \rho a^{n+1}}\left\{n^2 Y^{\frac{n-1}{n}}\frac{\partial^3 u^*}{\partial Y^3} + 2n\,(n-1)\frac{\partial^2 u^*}{\partial Y^2} Y^{-\frac{1}{n}} - (n-1)\,Y^{-\left(\frac{n+1}{n}\right)}\frac{\partial u^*}{\partial Y}\right\} \qquad (21)$$

where terms involving u^* or its derivatives to powers greater than 1 have been neglected. Equation (21) can be solved by separation of variables. One may set

$$u^* = U_0 \sum_i c_i \exp(-\lambda_i X)\,\varphi_i'(Y) \qquad (22)$$

where

$$X = \frac{U_0^{n-2}(1+2n)^{n-1} K x}{a^{n+1}\, n^n\, \rho} = \frac{(1+2n)^{n-1}}{n^n}\,\epsilon^{n+1} \qquad (23)$$

Substitution of Equation (22) into (21) results in a set of ordinary differential equations for the φ_i

$$n^2 Y^{\frac{n-1}{n}}\varphi_i'''' + 2n(n-1)\,Y^{-\frac{1}{n}}\varphi_i''' + (1-n)\,Y^{-\left(\frac{n+1}{n}\right)}\varphi_i'' + \left(\frac{2n+1}{n+1}\right)\left[1 - Y^{\frac{n+1}{n}}\right]\lambda_i\varphi_i'' + \left(\frac{2n+1}{n^2}\right) Y^{\frac{1-n}{n}}\lambda_i\varphi_i = 0 \qquad (24)$$

Symmetry with respect to the center line and the no-slip requirement at the wall result in boundary conditions

$$\varphi_i(0) = \varphi_i''(0) = 0$$

$$\varphi_i(1) = \varphi_i'(1) = 0$$

In addition, since the values of c_i are not yet fixed, $\varphi_i'(0)$ can have any nonzero value. It is convenient to set $\varphi_i'(0) = 1$. Solution of Equation (24) results in a set of eigenfunctions φ_i and corresponding eigenvalues λ_i for

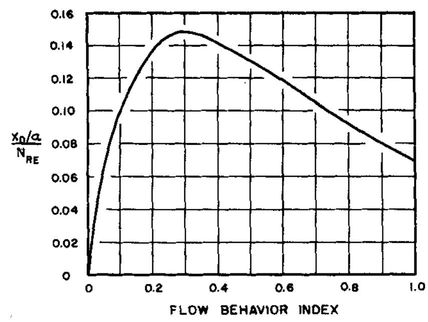

Fig. 3. Entry length as a function of flow behavior index.

* Appendix has been deposited as document 7354 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington 25, D. C., and may be obtained for $1.25 for photoprints or for 35-mm. microfilm.

which the boundary conditions are satisfied.

Values of c_i are those which will provide the best fit of the upstream solution to the downstream solution at the point where the two are joined. These values of c_i are found by minimizing the integral

$$\int_0^1 [U_0 \sum_i c_i \exp(-\lambda_i X_j) \varphi_i'(Y) - u_d^*]^2 dY \qquad (25)$$

where u_d^* is the value of the perturbation velocity determined from the downstream solution evaluated at the point where the solutions are joined. One proceeds by taking $(\partial)/(\partial c_i)$ of (25) and setting the result to zero. A set of simultaneous equations is obtained which permits evaluation of each of the c_i values.

PRESSURE LOSS

In the fully developed region of laminar flow in a channel it is readily shown that the change in pressure $|\Delta p|$ over a length Δx is given by

$$\frac{|\Delta p|}{\frac{1}{2}\rho U_0^2} = \frac{2^{n+1}\left(\frac{2n+1}{n}\right)^n}{N_{Re}} \frac{\Delta x}{a} \qquad (26)$$

The pressure loss between the entry to the channel where $x = 0$ (and $u =$ constant $= U_0$) and some point x_1 in the fully developed region is customarily given as a correction to the pressure loss computed from Equation (26). Thus the pressure loss between $x = 0$ and some point x_1 in the fully developed region may be expressed by

$$\frac{|\Delta p|}{\frac{1}{2}\rho U_0^2} = \frac{2^{n+1}\left(\frac{2n+1}{n}\right)^n}{N_{Re}} \frac{x_1}{a} + C \qquad (27)$$

where C is found from a momentum balance to be

$$C = 2\left(\frac{\overline{u^2}}{U_0^2} - 1\right) - \frac{2}{U_0^n}\left[\int_0^{x_d}\left|\frac{\partial u}{\partial Y}\right|^{n-1}\frac{\partial u}{\partial Y}\,dX\right]_{Y=1} - \frac{2^{n+1}\left(\frac{2n+1}{n}\right)^n}{N_{Re}}\frac{x_d}{a} \qquad (28)$$

The constant C represents the additional pressure loss occurring in the entry over and above that of fully developed flow.

It should be noted that all of the above equations consider the fluid to have a flat velocity profile $u = U_0$ at $x = 0$. Hence the change in pressure caused by acceleration of the fluid from a reservoir at $x < 0$ to the average channel velocity U_0 at $x = 0$ has not been included.

NUMERICAL PROCEDURE

Equations (12), (A1) through (A5), and (24) were solved with the Gill modification of the Runge-Kutta technique (8). Methods used to find initial conditions, when necessary, and the eigenvalues λ_i in Equation (24) are described elsewhere (5).

The downstream solution [Equation (16)] is a good approximation as long as $\delta^*/a << 1$ and Equation (11) is an accurate representation of $U(x)$. However Equations (11) and (16) diverge at sufficiently high values of ϵ. The downstream solution was considered valid only as long as the absolute value of each successive term in Equation (16) decreased. At the point where this ceased to be true, the downstream solution was joined to the upstream solution.

Equation (24) was solved for five values of n between 0.2 and 1.0. For each value of n three eigenvalues and the corresponding eigenfunctions were found. For example for $n = 0.8$ one finds

$$\lambda_1 = 15.7$$
$$\lambda_2 = 48.0$$
$$\lambda_3 = 97.9$$

Thus three terms of Equation (22) were used to express the upstream solution. Minimization of the expression (25) at the point where the upstream and downstream solutions were joined for $n = 0.8$ ($\epsilon_j = 0.0583$) results in

$$c_1 = 0.341$$
$$c_2 = -0.034$$
$$c_3 = 0.149$$

RESULTS

Calculations performed for $n = 0.2$, 0.4, 0.6, 0.8, and 1.0 form the basis for the results presented in Figures 1 through 6. Figure 1 shows the dependence of the constant C in Equation (27) on the flow behavior index n. For comparison a curve is also shown which gives the value of C obtained if one neglects viscous dissipation and equates the pressure loss in the entry to the change in kinetic energy of the fluid between $x = 0$ and $x = x_d$.

Since fully developed flow is approached asymptotically, one is required to choose an arbitrary criterion for the attainment of fully developed flow. In this work x_d was selected as the distance from the entry at which the center-line velocity had reached 98% of its fully developed value. A smooth curve with no sudden changes in slope is obtained if one plots ϵ_d vs. n, shown in Figure 2. The results are also presented in the form $(x_d)/(aN_{Re})$ vs. n in Figure 3. The latter method of presentation is analogous to the form usually employed for Newtonian fluids.

Figures 4 through 6 have been included to present a comparison of velocity profiles for different values of the flow behavior index at equivalent positions in the channel.

DISCUSSION

Equation (21), the upstream equation used to describe flow beyond the point at which Equation (16) is valid, was derived by neglecting products of u^* and its derivatives. When n is near unity, these terms are not always negligible in the vicinity of the point where the two solutions have been joined. The error thus incurred could be reduced by continuing the downstream solution to a higher value of ϵ and discarding terms in Equation (16) which have become divergent. This procedure was attempted for $n = 1$, with $\epsilon_j = 0.13$. Not only was the downstream solution somewhat less accurate because fewer terms were used, but correspondence between upstream and downstream solutions at the new ϵ_j was poor. In contrast using all six terms of Equation (16) and joining

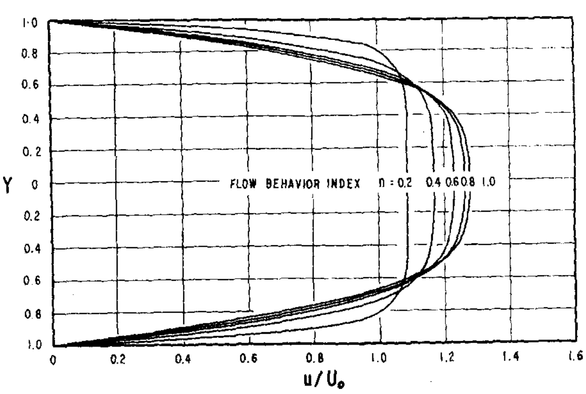

Fig. 4. Velocity profiles, $x = 0.25\,x_d$.

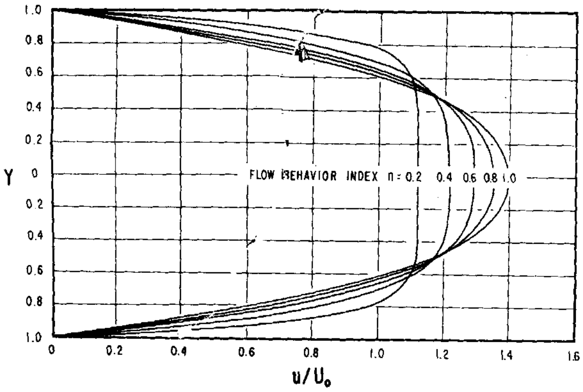

Fig. 5. Velocity profiles, $x = 0.50\ x_d$.

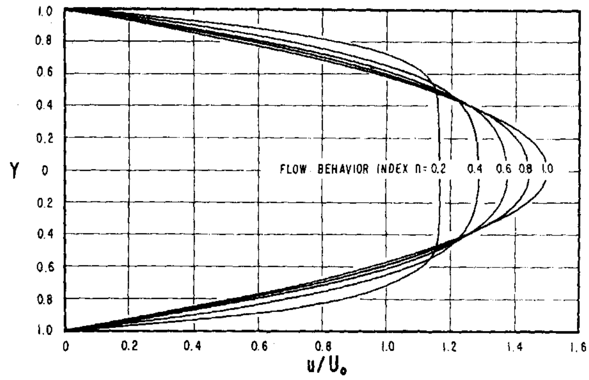

Fig. 6. Velocity profiles, fully developed flow.

the two solutions where Equation (16) begins to diverge ($\epsilon_j = 0.0709$ for $n = 1$) one obtains an excellent match of upstream and downstream solutions. Hence it is believed that the procedure used gives a reasonable estimate of entry behavior.

The values of entry length $(x_d)/(aN_{Re})$ and pressure-loss correction (C) for Newtonian fluids, which were found to be 0.069 and 0.74, respectively, are of interest. These compare with 0.080 and 0.60 reported by Schlichting (*10*). His downstream solution [Equation (16)] is composed of only three terms plus an approximation to the fourth. Also his upstream solution contains only one eigenvalue and consequently does not permit a close fit to the downstream solution at ϵ_j. The more accurate downstream solution used in the present study indicates that near the entry the drag force at the wall is somewhat higher than that obtained by Schlichting.

It is seen from Figure 3 that if one considers the flow of fluids with different values of n to be on an equivalent basis when the Reynolds numbers are identical, the entry length (in terms of the number of channel widths) actually increases as the flow behavior index decreases from 1 to about 0.3. It should be noted that if one chooses a Reynolds number equivalent to that used in this paper the curve of Bogue (*4*) showing $(x_d)/(aN_{Re})$ as a function of n for pipe flow, would also exhibit a maximum.

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NOTATION

a = channel half-width
C = constant defined by Equation (28)
c_i = constant defined by Equation (22)
$f_0, f_1, \ldots f_8$ = functions defined by Equations (12) and (A1) through (A5)
I_2 = second invariant of rate-of-strain tensor
K = consistency index defined by Equation (1)
K_i = constant defined by Equation (15)
N_{Re} = Reynolds number, $\dfrac{(2a)^n U_0^{2-n} \rho}{K}$
n = flow behavior index defined by Equation (1)
p = pressure
U = velocity outside of boundary layer
U_0 = average velocity in channel
u = x component of velocity
u_i = u, v, or w for $i = 1$, 2, or 3, respectively
u^* = velocity defined by Equation (20)
u_d^* = difference between u calculated from Equation (16) at $\epsilon = \epsilon_j$ and fully developed velocity
v = y component of velocity
w = z component of velocity
X = $\dfrac{(2n+1)^{n-1}}{n^n}\ \epsilon^{n+1}$
x = distance component in direction of fully developed flow
x_i = x, y, or z, for $i = 1$, 2, or 3, respectively
Y = $1 - y/a$
y = distance component perpendicular to channel walls
z = distance component perpendicular to xy plane

Greek Letters

δ = boundary-layer thickness
δ^* = displacement boundary-layer thickness
ϵ = $\left[\dfrac{Kx}{a^{n+1} U_0^{2-n} \rho}\right]^{\frac{1}{n+1}}$
η = $y\left[\dfrac{\rho U_0^{2-n}}{Kx}\right]^{\frac{1}{n+1}}$
η_1 = value of η outside of boundary layer
λ_i = eigenvalue defined by Equation (22)
μ_{eff} = effective viscosity
ρ = density
τ_{ij} = component of shear stress tensor defined by Equation (2)
φ_i = function defined by Equation (22)

Subscripts

d = evaluated at a value of x where center-line velocity is 98% of its fully developed value
j = evaluated at position where upstream and downstream solutions are joined

LITERATURE CITED

1. Acrivos, Andreas, M. J. Shah, and E. E. Petersen, *A.I.Ch.E. Journal*, **6**, 312 (1960).
2. Acrivos, Andreas, *ibid.*, p. 584.
3. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," pp. 101-104, Wiley, New York (1960).
4. Bogue, D. C., *Ind. Eng. Chem.*, **51**, 874 (1959).
5. Collins, Morton, Ph.D. thesis, Princeton Univ., Princeton, New Jersey (1962).
6. Hildebrand, F. B., "Introduction to Numerical Analysis," pp. 239-240, McGraw-Hill, New York (1956).
7. Oldroyd, J. G., in "Rheology," F. R. Eirich, ed., Vol. 1, pp. 653-682, Academic Press, New York (1956).
8. Romanelli, M. J., in "Mathematical Methods for Digital Computers," A. Ralston and H. S. Wilf, ed., Wiley, New York (1960).
9. Schowalter, W. R., *A.I.Ch.E. Journal*, **6**, 24 (1960).
10. Schlichting, Hermann, Z. *angew. Math. u. Mech.*, **14**, 368 (1934).
11. Tomita, Y., *Bull. Japan Soc. Mech. Engrs.*, **4**, 77 (1961).

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